# TRANSFORMATION OF A SYSTEM OF DIFFERENTIAL EQUATIONS OF HEAT AND MASS TRANSFER IN A DOMAIN WITH A VARIABLE BOUNDARY INTO A SYSTEM OF EQUATIONS FOR A DOMAIN WITH A FIXED BOUNDARY

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We present a method for transforming a system of differential equations of heat and mass transfer in a region with a variable boundary into an equivalent system of equations for a region with a fixed boundary.

The transport system of differential equations, together with its boundary conditions, represents in analytical form the basic features of the phenomena studied.

Solutions of the system enable us to obtain a picture of the distribution of transfer potentials in a body or system of bodies, to follow the variation of the fields of these potentials with time, and, based on this, to give a detailed analysis of the kinetics and the dynamics of the process involved.

A dimensionless mathematical model of heat and mass transfer is of the form

$$\frac{\partial T(x,\ \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{\partial^2 T(x,\ \mathrm{Fo})}{\partial x^2} + \frac{\Gamma}{x}\ \frac{\partial T(x,\ \mathrm{Fo})}{\partial x} - \mathrm{Ko}^*\ \frac{\partial \theta(x,\ \mathrm{Fo})}{\partial \mathrm{Fo}};$$
$$\frac{\partial \theta(x,\ \mathrm{Fo})}{\partial \mathrm{Fo}} = \mathrm{Ly}\left[\frac{\partial^2 \theta(x,\ \mathrm{Fo})}{\partial x^2} + \frac{\Gamma}{x}\ \frac{\partial \theta(x,\ \mathrm{Fo})}{\partial x}\right] - \mathrm{Ly\,Pn}\left[\frac{\partial^2 T(x,\ \mathrm{Fo})}{\partial x^2} + \frac{\Gamma}{x}\ \frac{\partial T(x,\ \mathrm{Fo})}{\partial x}\right]. \tag{1}$$

Analytical solutions of this system are presented in considerable detail in [1]. The heat- and mass-transfer problems treated in [1] involved bodies whose dimensions stayed the same as the process proceeded.

Besides these problems there are a number of engineering problems of heat and mass transfer in which the dimensions of the bodies vary. Thus, in the process of drying, a change is observed in the linear dimensions of a body, wherein, subject to small gradients in moisture content and temperature, there is a shrinkage of the material in accordance with a linear behavior; in more rigid drying modes, however, large gradients of moisture content and temperature arise in a material and the resulting shrinkage follows a much more involved nonlinear variation.

There are also heat-conduction problems in which it is necessary to account for the change in the linear dimensions of a body. Thus, there are problems involving a phase transition, i.e., problems involving moving boundaries, for example, Stefan problems.

We present here a method for solving a system of differential equations of heat and mass transfer with expanding or contracting boundaries.

In the system (1) let $x \in [R(\mathrm{Fo}),\ \infty)$, where $R(\mathrm{Fo})$ is a known function which defines the way the surface is being displaced; we assume this function to be continuous and to have continuous first and second derivatives.

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As was shown in [3], the system (1) can be reduced to a system of two equations, similar in form to the thermal-conductivity equation for a combination function $Z_i$:

$$\frac{\partial Z_i}{\partial \mathrm{Fo}} = \frac{1}{\mu_i^2} \left( \frac{\partial^2 Z_i}{\partial x^2} + \frac{\Gamma}{x} \frac{\partial Z_i}{\partial x} \right), \tag{2}$$

where $Z_i = m_i T(x, \mathrm{Fo}) + n_i \theta(x, \mathrm{Fo}) (i = 1, 2)$, and the numbers $\mu_i$ are obtained from the expressions

$$\mu_i^2 = \frac{1}{2} \left[ \left( 1 + \mathrm{PnKo}^* + \frac{1}{\mathrm{Ly}} \right) + (-1)^i \sqrt{\left( 1 + \mathrm{PnKo}^* + \frac{1}{\mathrm{Ly}} \right)^2 - \frac{4}{\mathrm{Ly}}} \right].$$

The dimensionless numbers $m_i$ and $n_i$ will be completely defined if we put $n_i = 1$, since

$$\frac{m_i}{n_i} = \frac{\mathrm{Pn}}{\mu_i^2 - 1}.$$

We can apply Grinberg's method (see [2]) to the equations of the system (2).

We put $y = x/R(\mathrm{Fo})$, where $y \in [1, \infty)$. This transformation transforms the varying domain $[R(\mathrm{Fo}), \infty)$ to the fixed domain $[1, \infty)$. The system (2) then assumes the form

$$\frac{\partial Z_i}{\partial \mathrm{Fo}} = \frac{\partial Z_i}{\partial y} \frac{\dot{R} y}{R} + \frac{1}{\mu_i^2 R^2} \left[ \frac{\partial^2 Z_i}{\partial y^2} + \frac{\Gamma}{y} \frac{\partial Z_i}{\partial y} \right], \tag{3}$$

where $R = R(\mathrm{Fo})$, $\dot{R} = dR(\mathrm{Fo})/d\mathrm{Fo}$. By making the substitution

$$Z_i(y, \mathrm{Fo}) = q_i(y, \mathrm{Fo}) \cdot V_i(y, \mathrm{Fo})$$

we go from the system of equations (3), containing the derivative of the unknown function with a coefficient depending on the time and a coordinate, to equations in the new functions $V_i(y, \mathrm{Fo})$ not containing this term. After a substitution and some simplifications (see [2]), we obtain

$$\frac{1}{\mu_i^2} \left[ \frac{\partial^2 V_i}{\partial y^2} + \frac{\Gamma}{y} \frac{\partial V_i}{\partial y} \right] + \frac{y^2 \mu_i^2 R^3 \ddot{R}}{4} V_i = R^2 \frac{\partial V_i}{\partial \mathrm{Fo}}, \tag{4}$$

$$q_i = R^{-\frac{1+\Gamma}{2}} \exp \left( -\frac{y^2 \mu_i^2 R \dot{R}}{4} \right). \tag{5}$$

The equations appearing in the system (4) can be solved in the case when $R^3 \ddot{R} = \mathrm{const}$, i.e., when the motion of the boundary follows the law

$$R(\mathrm{Fo}) = \sqrt{M \mathrm{Fo}^2 + N \mathrm{Fo} + P}, \tag{6}$$

where M, N, and P are constants.

Consequently, the corresponding boundary-value problem for the initial system (1) may be solved for domains of this form but with dimensions varying in accordance with the law indicated. In particular, if $R^3 \ddot{R} = 0$, i.e., if the boundary of the domain moves in accordance with the linear law

$$R(\mathrm{Fo}) = A \mathrm{Fo} + B, \tag{7}$$

where A and B are constants, we obtain the system

$$\frac{\partial^2 V_i}{\partial y^2} + \frac{\Gamma}{y} \frac{\partial V_i}{\partial y} = \mu_i^2 R^2 \frac{\partial V_i}{\partial \mathrm{Fo}} \tag{8}$$

The expressions (8) and (4) are similar to the thermal-conductivity equation in which the coeffficient of thermal conductivity is time dependent.

Through the substitution $d\tau = d\mathrm{Fo}/R^2(\mathrm{Fo})$ such equations may, as is well known, be reduced to the thermal-conductivity equation with a constant coefficient of thermal conductivity, solution methods for which are known.

Transformation of Eqs. (1) and (2) for the case in which $x \in [0, R(\mathrm{Fo})]$ again leads to the system (3) for the domain $y \in [0, 1]$. The subsequent considerations apply then to this case also.

The laws describing the motion of the domain boundary, given by the relations (6) and (7), encompass a sufficiently broad collection of engineering heat- and mass-transfer problems.

It should be remarked that we can also apply the method in question to the system (1), wherein the latter is augmented by heat and matter sources; we can also apply it to a system containing, not two, but n transfer potentials.

## NOTATION

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| T | is the temperature; |
| $\theta$ | is the moisture content; |
| Fo | is the Fourier number; |
| Ko, Ly, Pn | are the Kossovich, Lykov, and Posnov numbers, respectively; |
| $Ko^* = \varepsilon Ko$, | where $\varepsilon$ is the factor of phase transition of a liquid into a vapor; |
| $\Gamma = 0, 1, 2$ | for a plate, cylinder, and sphere, respectively. |

## LITERATURE CITED

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# NUMERICAL ALGORITHM OF THE SOLUTION OF THE MULTIPHASE STEFAN PROBLEM

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A method is proposed for computing the temperature and position of the phase interface based on the passage to new variables and a new function. The transformation is invariant relative to the heat-conduction equation, and the boundaries in the new variables are fixed.

A whole series of papers on the Stefan problem exist, which are surveyed sufficiently completely in [1], and wherein a great deal of original material associated with the proof of the uniqueness and existence of the solution is also generalized. Numerical schemes for the solution are proposed in [2]. Significant attention is paid there to the mathematical aspect of the question, but no results are presented of practical tests or of computations. V. G. Melamed [3] also gave a numerical solution, realized in application to the case of freezing soils. Fundamental results of a cycle of the author's work are presented in [3]. An analogous problem in terms of physical content, but taking account of snow and the influence of the atmosphere, is considered in [4]. Let us note that the nature of the method of solution to be used is determined by the specifics of some definite problem to be solved, which is a particular case of the general Stefan problem. The present paper, which is oriented toward the hydrometeorology area from the viewpoint of practical applications, is organized in a similar plan.

We formulate the problem below. Let us examine the one-dimensional case. Between two fixed planes $z = 0$ and $z = H$ at a time $t = 0$ let there be n alternating layers of material in the liquid or solid aggregate state with the moving interfaces $z = h_m(t)$ $(m = 1, 2, \ldots, n-1)$, where phase transition occurs. Let one layer of another material whose outer boundary moves according to the known law $z = -l(t)$ also adjoin the surface $z = 0$. The initial temperature distribution is given in the whole domain $T^0(z)$. Let us consider the temperature a known

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